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The V–N Sector of the Lee Model

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Abstract

The V–N sector of a modified Lee model is solved by dispersion theory techniques. The method of solution clearly indicates the importance of asymptotic conditions in solving bound-state problems.

1. *Introduction*

The $V-N$ sector of the Lee model (Lee, 1954) has been considered by Weinberg (1955) and from the viewpoint of dispersion theory of Scarfone (1962). In this paper we re-examine the problem of determining the eigenvalue condition for the $V+N$ potential energy using dispersion theory. We show that the problem can be solved without having to consider two vertex functions $\Gamma \equiv \langle V|f_N|B\rangle$ and $\Gamma' \equiv \langle N|f_V|B\rangle$ as done by Scarfone. The interesting point of our method is that it clearly shows the importance of asymptotic conditions in bound-state problems.

It will be recalled that the Lee model describes the coupling of two infinitely heavy fermions V and N with a relativistic boson such that $V \rightleftarrows N+\theta$ is the basic interaction. Following Scarfone we take the $V+N$ separation to be zero and assume that all field operators satisfy commutation relations. In Section 2 we describe the model and sketch the normal simple method of solving the $V+N$ energy eigenvalue problem. In Section 3 we give the dispersion theory method.

2. *Eigenvalue Condition by Standard Methods*

The Hamiltonian which describes the Lee model is given by

$$H=(m+\delta m)Z\psi_V{}^+\psi_V+m\psi_N{}^+\psi_N+\sum_k \omega a_k{}^+a_k+g\psi_V{}^+\psi_N+gA^+\psi_N{}^+\psi_V \tag{2.1}$$

where

$$A = \sum_k \frac{u(\omega)}{(2\omega\Omega)^{1/2}} a_k \qquad [\omega = (k^2 + \mu^2)^{1/2}] \tag{2.2}$$

The cut-off function $u(\omega)$ assures the convergence of all integrals and is chosen such that the theory contains no ghost effects. The operator ψ_V is the renormalised V particle field operator, g is the renormalised coupling constant and Z is a renormalisation constant. It is convenient to assume that V and N have the same mass. As usual, we are quantising in a box of volume Ω which eventually becomes infinite. Throughout this paper all field operators will obey commutation relations

$$[a_{k'}, a_k^+] = \delta_{k,k'}, \qquad [\psi_N, \psi_N^+] = 1, \qquad [\psi_V, \psi_V^+] = \frac{1}{Z}[a_k, a_N]$$
$$= [\psi_N, \psi_N] = [\psi_V, \psi_V] = 0 \tag{2.3}$$

Having chosen Z such that $\langle 0|\psi_V| V\rangle = 1$, the eigenvalue problem $H| V\rangle = m|V\rangle$ yields

$$\delta m = \frac{g^2}{2Z\Omega} \sum_k \frac{u^2(\omega)}{\omega^3} \tag{2.4}$$

$$Z = 1 - \frac{g^2}{2\Omega} \sum_k \frac{u^2(\omega)}{\omega^2} \tag{2.5}$$

These results are also obtainable by dispersion methods (DeCelles & Feldman, 1959). We review, following Scarfone (1962), the discrete $V + N$ spectrum of the above model. The corresponding problem in the ordinary Lee theory with finite heavy particle separation has been studied by Weinberg (1955).

We know from the selection rules that $|B\rangle$ has the following mixture of bare states

$$|B\rangle = ||V, |N\rangle + \sum_k \phi(k)\, |2N, |k\rangle \tag{2.6}$$

where

$$||V, |N\rangle = \psi_V^+ \psi_N^+ |0\rangle$$
$$|2N, |k\rangle = \frac{1}{\sqrt{2}} \psi_N^+ \psi_N^+ a_k^+ |0\rangle$$

We observe here that the Bose character of the heavy particle operator permits us to have more than one of them at the same point in a given state. This is the motivation behind the commutation relations (2.3). Calling the eigenvalue of the state (2.6) $(Zm + \omega_0)$ and using (2.1) to (2.4) we obtain

$$\omega_0 = \delta m + g\sqrt{2} \sum_k \frac{u(\omega)\, \phi(k)}{(2\omega\Omega)^{1/2}} \tag{2.7}$$

$$(\omega - \omega_0)\, \phi|k\rangle = -\frac{g}{Z}\sqrt{2}\, \frac{u(\omega)}{(2\omega\Omega)^{1/2}} \tag{2.8}$$

The factor $(\omega-\omega_0)$ in (2.8), where ω is the energy of the light particle and ω_0 the potential energy of interaction (recall there are no recoil effects), does not vanish, since we are only concerned with bound states. Eliminating $\phi(k)$ from (2.7) and (2.8) we find

$$\omega_0-\delta_m=-\frac{g^2}{Z\Omega}\sum_k\frac{u^2(\omega)}{\omega}\frac{1}{\omega-\omega_0} \tag{2.9}$$

for the determination of the eigenvalue ω_0. With the help of (2.4) and (2.5) we can put (2.9) into the form

$$1-\beta(\omega_0)=-\lambda(\omega_0) \tag{2.10}$$

where in general

$$\beta(\omega)\equiv-\frac{g^2}{2\Omega}\omega\sum_k\frac{u^2(\omega)}{\omega'^3(\omega'-\omega-i\varepsilon)} \tag{2.11}$$

$$\lambda(\omega_0)\equiv\frac{g^2}{2\Omega\omega_0}\sum_k\frac{u^2(\omega)}{(\omega-\omega_0)} \tag{2.12}$$

Equation (10) is the desired condition which gives ω_0 as a function of the renormalised coupling constant and the cut-off functions. We now proceed to show that it is possible to reproduce condition (2.10) without knowing the state vectors.

3. *Eigenvalue Condition by Dispersion Methods*

We consider the vertex function†

$$\Gamma\equiv\langle V|g_N(0)|B\rangle \tag{3.1}$$

where from the Heisenberg equation of motion for ψ_N we have

$$g_N(t)=\left(-i\frac{d}{dt}+m\right)\psi_N(t)=-gA^+(t)\psi_V(t) \tag{3.2}$$

If the V-particle is contracted in (3.1) we are led to the expression

$$\Gamma=i\int_{-\infty}^{+\infty}dt\exp(imt)\langle 0|[g_V(t),g_N(0)]\theta|t)|B\rangle \tag{3.3}$$

where from the Heisenberg equation of motion for ψ_V we have

$$g_V(t)\equiv\left(-i\frac{d}{dt}+m\right)\psi_V(t)=-\delta m\psi_V(t)-\frac{g}{2}\psi_N(t)A(t) \tag{3.4}$$

while the equal time commutators resulting from the differentiation of the theta function gives zero. If a complete set of intermediate states is inserted

† This approach of considering vertex functions for investigating the properties of a composite particle is due to Blankenbecker, R. and Cook, L. F. (1960). *Physical Review*, **119**, 1745.

into (3.3) and the time integration performed after subjecting $f(t)$ to a time translation we arrive at

$$\Gamma = \sum_{S} \frac{\langle\langle 0|g_V|S\rangle\langle S|g_N|B\rangle}{E_S - m - i\varepsilon} \tag{3.5}$$

The second term of the commutator vanishes, since only $|S\rangle = |N\rangle$ contributes, but $\langle 0|g_N|N\rangle = 0$. Now the S states must have the same quantum number as a V-particle, and since $\langle 0|g_V|U\rangle = 0$ we are left with the $N+\theta$ scattering states. We have then

$$\Gamma = \sum_{k} \frac{1}{\omega}\langle 0|g_V|N\theta_\omega\rangle\langle N\theta_\omega|g_N|B\rangle \tag{3.6}$$

where we shall take $|N\theta_\omega\rangle$ to mean 'out' states. We observe that two new matrix elements occur, and we define

$$Q(\omega) \equiv \frac{(2\omega\Omega)^{1/2}}{u(\omega)}\langle 0|g_V(0)|N\theta_\omega\rangle \tag{3.7}$$

and

$$R(\omega) \equiv \frac{(2\omega\Omega)^{1/2}}{u(\omega)}\langle N\theta_\omega|g_N(0)|B\rangle \tag{3.8}$$

Let us consider $R(\omega)$. We contract the N-particle and with the help of the definition of the asymptotic states arrive at the expression

$$R(\omega) = \frac{(2\omega\Omega)^{1/2}}{u(\omega)} \int_{-\infty}^{+\infty} \exp(imt)\, dt\langle\theta_\omega|[g_N(t), g_N(0)]\,\theta(t)|B\rangle \tag{3.9}$$

where the equal-time commutator vanishes. Inserting a complete set of states and performing the time integrals given

$$R(\omega) = -\frac{(2\omega\Omega)^{1/2}}{u(\omega)}\langle\theta_\omega|g_N|V\rangle\langle V|g_N|B\rangle\left[\frac{1}{\omega}+\frac{1}{\omega_0}\right] + \frac{(2\omega\Omega)^{1/2}}{u(\omega)}\sum_{k'}\langle\theta_\omega|g_N|N\theta_{\omega'}\rangle\langle N\theta_{\omega'}|g_N|B\rangle\left[\frac{1}{\omega'-\omega-i\varepsilon}+\frac{1}{\omega'-\omega_0}\right] \tag{3.10}$$

Since

$$(-)\frac{(2\omega\Omega)^{1/2}}{u(\omega)}\langle\theta_\omega|g_N(0)|V\rangle = g$$

and

$$\langle V|g_N(0)|B\rangle = \Gamma$$

we can write $R(\omega)$ as

$$R(\omega) = g\Gamma\left[\frac{1}{\omega}+\frac{1}{\omega_0}\right] + \frac{(2\omega\Omega)^{1/2}}{u(\omega)}\sum_{k'}\frac{u(\omega')}{(2\omega'\Omega)^{1/2}}\langle\theta_\omega|g_N(0)|N\theta_{\omega'}]\,R(\omega')\left[\frac{1}{\omega'-\omega-i\varepsilon}+\frac{1}{\omega'-\omega_0}\right] \tag{3.11}$$

The matrix element $\langle\theta_\omega|g_V(0)|N\theta_{\omega'}\rangle$ can be converted into known functions and to do this we define

$$D(\omega') \equiv \frac{(2\omega\Omega)^{1/2}}{u(\omega)}\langle\theta_\omega|g_N(0)|N\theta_{\omega'}\rangle \tag{3.12}$$

Contracting the θ-particle in the left we are led to the expansion

$$D(\omega') = i\int_{-\infty}^{+\infty} dt\exp(i\omega t)\langle 0|[J(t), g_N(0)]\,\theta(t)|N\theta_{\omega'}\rangle - g\langle 0|\psi_V|N\theta_{\omega'}\rangle \tag{3.13}$$

where

$$j(t) = \left(-i\frac{d}{dt}+\omega\right)a_k(t)\cdot\frac{(2\omega\Omega)^{1/2}}{u(\omega)} = -g\psi_N^{+}(t)\,\psi_V(t)$$

and the second term comes from the equal time commutator. Inserting a complete set of states in the first term and carrying out the time integration we find that both parts of the commutator vanish, since only one-particle states contribute. The second term can be related to $N+\theta$ scattering phase shifts. Indeed,

$$D(\omega) = \frac{u(\omega)}{(2\omega\Omega)^{1/2}}\left[4\pi\frac{\exp[-i\delta(\omega)]\sin\delta(\omega)}{u^2(\omega)(\omega^2-\mu^2)}\right] \tag{3.14}$$

Expansions for $\delta(\omega)$ in terms of the function $\beta(\omega)$ are known (Goldberger & Treiman, 1959). Letting $\Omega\to\omega$ and using (3.14), (3.11) becomes:

$$R(\omega) =$$

$$g\Gamma\left[\frac{1}{\omega}+\frac{1}{\omega_0}\right]+\frac{1}{\pi}\int_\mu^\infty \exp[-i\delta(\omega')]\sin\delta(\omega')\,R(\omega')\left[\frac{1}{\omega'-\omega-i\varepsilon}+\frac{1}{\omega'-\omega_0}\right]d\omega' \tag{3.15}$$

We suppose $\omega_0 < 0$. We now proceed to solve this integral equation. Introduce

$$\exp[\Delta(z)]\,F(z) = \frac{i}{2\pi i}\int_\mu^\infty d\omega'\exp[-i\delta(\omega')]\sin\delta(\omega')\,R(\omega')\left[\frac{1}{\omega'-z}+\frac{1}{\omega'-\omega_0}\right] \tag{3.16}$$

where

$$\Delta(z) = \frac{1}{\pi}\int_\mu^\infty d\omega'\exp[i\delta(\omega')]\left[\frac{1}{\omega'-z}+\frac{1}{\omega'-\omega_0}\right] \tag{3.17}$$

and

$$\delta(\omega) = (-)\frac{1}{2i}\ln\left[\frac{1-\beta(\omega)}{1-\beta^*(\omega)}\right] \tag{3.18}$$

Define

$$\Delta_\pm(\omega) = \Delta(\omega\pm i\varepsilon), \qquad F_\pm(\omega) = F(\omega\pm i\varepsilon)$$

Then

$$\exp[+(\omega)]F_+(\omega)-\exp[\Delta-(\omega)]F-(\omega)=\exp[i\delta(\omega)]\sin\delta(\omega)R(\omega) \quad (3.19)$$

and

$$R(\omega)=g\Gamma\left[\frac{1}{\omega}+\frac{1}{\omega_0}\right]+2i\exp[\Delta_+(\omega)]F_+(\omega) \quad (3.20)$$

Thus

$$\exp[\Delta_+(\omega)]F_+(\omega)-\exp[\Delta_-(\omega)]F_-(\omega)=\exp[-i\delta(\omega)]\sin\delta(\omega)g\Gamma\left[\frac{1}{\omega}+\frac{1}{\omega_0}\right]$$
$$+\left[\frac{1-\exp[-2i\delta(\omega)]}{2i}\right]2i\exp[\Delta_+(\omega)]F_+(\omega)$$

This gives

$$F_+(\omega)-F-(\omega)=\exp[-\delta(\omega)]\sin\delta(\omega)\left[\frac{1}{\omega}+\frac{1}{\omega_0}\right] \quad (3.21)$$

where

$$\rho(\omega)+i\delta(\omega)=\frac{1}{\pi}\int_{\mu}^{\infty}dx\,\delta(u)\left[\frac{1}{u-\omega'-i\varepsilon}+\frac{1}{x+\omega'-\omega}\right] \quad (3.22)$$

Thus

$$F(z)=g\Gamma\left\{A+\frac{1}{2\pi i}\int_{\mu}^{\infty}d\omega'\exp[-\rho(\omega')]\sin\delta(\omega')\left[\frac{1}{\omega'}+\frac{1}{\omega_0}\right]\left[\frac{1}{\omega'-z}+\frac{1}{\omega'-\omega_0}\right]\right\}$$
$$(3.23)$$

The arbitrary constant A is to be so selected that the asymptotic behaviour of $F(z)$ obtained from (3.23) is the same as that obtained from (28).† Using (2.23) we see that

$$2iF_+(\omega)=g\Gamma\left\{2i\,A_+\left(\frac{1-\beta(\omega_0)}{Z^2}\left[\beta(\omega)\left(\frac{1}{\omega_0}+\frac{1}{\omega}\right)+\frac{2}{\omega_0}(\beta(\omega_0)-1+Z\right]\right)\right\}$$

So that

$$R(\omega)=\frac{g\Gamma}{1-\beta(\omega)}\left[\frac{1}{\omega}+\frac{K}{\omega_0}\right] \quad (3.24)$$

where

$$K=1-2(\beta(\omega_0)-1)+2Z+\left(\frac{Z^2}{1-\beta(\omega_0)}\right)2iA\omega_0$$

Substituting this expression in (3.16) and remembering that

$$\exp[\Delta_+(\infty)]=\left[\frac{Z}{1-\beta(\omega_0)}\right] \quad \text{and} \quad 1-\beta(\omega)\to Z,\ |\omega|\to\infty$$

† I would like to thank Prof. J. Bronzan for pointing out the importance of the constant A.

by taking the limit $|\omega| \to \infty$ and equating the results for $[F_+(\omega)]_{|\omega|\to\infty}$ from equations (3.16) and (3.23) we get the equation

$$g\Gamma(2iA)\left[1-\left(\frac{1}{\omega_0 Z}-\frac{1-\beta(\omega_0)}{\omega_0 Z^2}\right)\frac{Z^2\omega_0}{1-\beta(\omega_0)}\right]$$
$$=-g\Gamma\frac{1-\beta(\omega_0)}{Z^2}\left(\frac{1-Z}{\omega_0}+\frac{2}{\omega_0}(\beta(\omega_0)-1+Z)\right)$$
$$+g\Gamma\left\{\frac{\beta(\omega_0)}{\omega_0 Z}+\left(\frac{1}{\omega_0 Z}-\frac{1-\beta(\omega_0)}{\omega_0 Z^2}\right)(1+2(\beta(\omega_0)-1)+2Z)\right\}$$

which determines A. Substituting this in (3.24) gives

$$R(\omega)=\frac{g}{1-\beta(\omega)}\left[\frac{1}{\omega}+\frac{1}{\omega_0}\frac{Z}{2(1-\beta(\omega_0))-Z}\right] \tag{3.25}$$

The matrix element $Q(\omega)$ in (3.7) has been evaluated by Goldberger & Treiman (1959), and their result is

$$Q(\omega)-(-)\left[\frac{g}{1-\beta(\omega)}\right]^* \tag{3.26}$$

Rewriting (3.6) as (in the limit $\Omega\to\infty$)

$$\Gamma=\frac{1}{4\pi^2}d\omega\frac{u^2(\omega)(\omega^2-\mu^2)^{1/2}}{\omega}0(\omega)R(\omega)$$

and using (3.25) and (3.26) gives

$$1=\frac{1}{\pi}\int_{\mu}^{\infty}d\omega\,\mathrm{Im}\left[\frac{1}{1-\beta(\omega)}\right]\left[\frac{1}{\omega}+\frac{1}{\omega_0}\right]$$

which is equivalent to

$$1-\beta(\omega_0)=-\lambda(\omega_0)$$

as can be easily verified.

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